2) Gauss

$\psi(r) = N' \cdot e^{-\beta r^2}$

$\langle\psi|\psi\rangle = 1$

$$\Rightarrow N'^2 \cdot 4\pi \underbrace{\int_0^\infty dr\, r^2 \cdot e^{-2\beta r^2}}_{-\frac{1}{2}\cdot\frac{\partial}{\partial\beta}\underbrace{\int_0^\infty e^{-2\beta r^2}\, dr}_{\frac{1}{2}\cdot\sqrt{\frac{\pi}{2\beta}}}} = N'^2 \cdot 4\pi \left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{1}{2}\right)\sqrt{\frac{\pi}{2}}\,\beta^{-3/2} =$$

$$= N'^2 \left(\frac{\pi}{2\beta}\right)^{3/2} = 1 \qquad \Rightarrow \quad N' = \left(\frac{2\beta}{\pi}\right)^{3/4}$$

$$\langle E_{pot}\rangle = \langle\psi| -\frac{Z}{r} |\psi\rangle = -Z N'^2 \cdot 4\pi \int_0^\infty dr\, r^2 e^{-\beta r^2} \cdot \frac{1}{r} e^{-\beta r^2} =$$

$$= -Z N'^2 \cdot 4\pi \cdot \int_0^\infty dr \cdot \underbrace{r \cdot e^{-2\beta r^2}}_{\frac{1}{-4\beta}\frac{\partial}{\partial r}\left(e^{-2\beta r^2}\right)} = -Z N'^2 \cdot 4\pi \cdot \frac{-1}{4\beta}\left[e^{-2\beta r^2}\right]_0^\infty = \underline{-Z\sqrt{\frac{8\beta}{\pi}}}$$

$$\langle E_{kin}\rangle = \langle\psi| -\frac{1}{2}\Delta |\psi\rangle = -\frac{1}{2} N'^2 \cdot 4\pi \int_0^\infty dr\, \cancel{r^2} e^{-\beta r^2}\left[\frac{1}{\cancel{r^2}}\frac{\partial}{\partial r} r^2 \underbrace{\frac{\partial}{\partial r} e^{-\beta r^2}}_{-2\beta r\, e^{-\beta r^2}}\right] =$$

$$= -\frac{1}{2} N'^2 \cdot 4\pi (-2\beta) \int_0^\infty dr\, e^{-\beta r^2}\left[\underbrace{3r^2 e^{-\beta r^2}}_{3\left(-\frac{1}{2}\right)\frac{\partial}{\partial\beta}\left(e^{-2\beta r^2}\right)} + \underbrace{r^3(-2\beta r)e^{-\beta r^2}}_{-2\beta\cdot\frac{1}{4}\frac{\partial^2}{\partial\beta^2}\left(e^{-2\beta r^2}\right)}\right] =$$

$$= -\frac{1}{2} N'^2 \cdot 4\pi (-2\beta)\left(-\frac{1}{2}\right)\left[3\underbrace{\frac{\partial}{\partial\beta}\left(\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\right)}_{3\cdot\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\beta^{-3/2}} + \underbrace{\beta\frac{\partial^2}{\partial\beta^2}\left(\frac{1}{2}\sqrt{\frac{\pi}{2\beta}}\right)}_{\beta\cdot\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\beta^{-5/2}}\right] =$$

$$= -\frac{1}{2}\,\frac{2\beta}{\pi}\sqrt{\frac{2\beta}{\pi}} \cdot 4\pi \cdot \beta \cdot \left(-\frac{3}{4}\right)\sqrt{\frac{\pi}{2}} \cdot \beta^{-3/2} \cdot \left[1 - \frac{1}{2}\right] = \underline{\frac{3}{2}\beta}$$

$$\Rightarrow \langle E\rangle = \frac{3}{2}\beta - Z\sqrt{\frac{8\beta}{\pi}}$$

$$0 = \frac{\partial\langle E\rangle}{\partial\beta} = \frac{3}{2} - Z\sqrt{\frac{8}{\pi}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\beta}} \quad \Rightarrow \quad \underline{\beta_0 = \frac{8 \cdot Z^2}{9\pi}}$$

$$\Rightarrow E_0 = \frac{3}{2} \cdot \frac{8 \cdot Z^2}{9\pi} - Z \cdot \frac{8 \cdot Z}{3\pi} = \underline{-\frac{4 \cdot Z^2}{3\pi}}$$

Ha $Z = 1 \rightarrow E_0 = -0{,}424$

Megj: 3, 4, … 6 db Gauss $\rightarrow E = -0{,}499 > -\frac{1}{2}$

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